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Asymmetric one-dimensional exclusion processes: a two-parameter discrete-time exactly solvable model

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Abstract
A two-parameter family of discrete-time asymmetric exclusion processes on a one-dimensional lattice is considered. Among these processes are the asymmetric simple exclusion model and the drop–push model. Using the Bethe ansatz, the exact solution to the master equation and from that the drift rates of the two-particle sector are obtained.

1. Introduction

A reaction–diffusion system consists of a collection of particles (of one or several species) moving and interacting with each other with specific probabilities (or rates in the case of a continuous time variable). In the so-called exclusion processes, any site of the lattice that the particles move on is either vacant or occupied by one particle. The aim of studying such systems is of course to calculate the time evolution of the systems.

Various classes of reaction–diffusion systems are called exactly solvable, in different senses. In [1–3], for example, integrability means that the N-particle conditional probabilities' S-matrix is factorized into a product of two-particle S-matrices, while in [4–10], solvability means closedness of the evolution equation of the empty intervals (or their generalization). Here we consider discrete-time asymmetric exclusion processes in a one-dimensional lattice, which are solvable in the first sense.

In [11], a method was introduced to solve some solvable systems (in the first sense mentioned above). In that method, one replaces the interaction of the particles with suitable boundary conditions, and finds the Bethe-ansatz solution, which consists of a linear combination of plane waves. From these, one can construct the conditional probability of finding particles on lattice sites. This method was used in [12] and [13] to investigate a two-parameter family of exclusion models, containing the simple exclusion model and the drop–push model on a one-dimensional lattice. In [14], this method was used to investigate the same family on a continuum.

The model that we are considering here consists of a one-dimensional lattice every site of which is either empty or occupied by a single particle. In this view it resembles the models introduced above. However, here the evolution parameter (time) is taken to be discrete, and there remain different possibilities for updating the configuration of the system. One possibility is sequential updating, and another possibility is simultaneous updating. In the first case, at each evolution step the position of at most one of the particles is updated (if the particles are not adjacent), while in the second case the positions of more than one particle can be updated simultaneously. The particles hop to the right with a certain probability if their right-hand-side site is empty. One can see that if this probability is small, then the difference between the above two possibilities is removed. If this is not the case, however, then the above two possibilities lead to different results.

Here we consider a discrete-time exclusion reaction–diffusion model on a one-dimensional lattice, choose simultaneous updating, and impose relations between reaction rates so that the system is solvable in the sense of the factorizability of the S-matrix. The conditional probability is obtained, and from it the drift rates in the two-particle sector are calculated. The results are compared with the already known results for the continuous-time evolution version.

The scheme of the paper is the following. In section 2, the evolution equation is obtained and it is shown that the interaction between the particles can be replaced by a suitable boundary condition. In section 3, the conditional probability and drift rates for the two-particle sector are calculated. Section 4 is devoted to the concluding remarks.

2. Discrete-time one-dimensional asymmetric exclusion processes

Consider a one-dimensional lattice, each site of which is either empty or occupied by one particle. The probability that the first particle is in x_1, the second particle is in x_2, etc, is

$$P(x_1, x_2, \ldots), \qquad x_1 < x_2 < \cdots .$$

The process is that each particle can hop to the right, with the probability α, if the its right-hand-side neighbour is empty:

$$A\emptyset \to \emptyset A, \qquad \text{with the probability } \alpha.$$

Consider the following evolution equation and boundary condition for the two-particle sector:

$$\begin{aligned} P(x_1, x_2, t+1) = {} & (1-\alpha)^2 P(x_1, x_2, t) + \alpha(1-\alpha)[P(x_1-1, x_2, t) + P(x_1, x_2-1, t)] \\ & + \alpha^2 P(x_1-1, x_2-1, t), \qquad x_1 < x_2, \end{aligned} \tag{1}$$

and

$$P(x, x) = \lambda P(x, x+1) + \mu P(x-1, x), \qquad \lambda + \mu = 1. \tag{2}$$

These describe a system where particles can push:

$$AA\emptyset \to \emptyset AA, \qquad \text{with the probability } \beta$$

where

$$\beta = \alpha(1-\alpha)\mu + \alpha^2 = \alpha - \alpha(1-\alpha)\lambda. \tag{3}$$

3. The two-particle sector conditional probability, and the drift rates

Equation (1) can be written as

$$P(\mathbf{x}, t+1) = U P(\mathbf{x}, t). \tag{4}$$

Then the Bethe-ansatz solution corresponding to the eigenvalue u is

$$u\Psi(x_1, x_2) = (1-\alpha)^2 \Psi(x_1, x_2) + \alpha(1-\alpha)[\Psi(x_1 - 1, x_2) + \Psi(x_1, x_2 - 1)] + \alpha^2 \Psi(x_1 - 1, x_2 - 1) \tag{5}$$

and

$$\Psi(x, x) = \lambda\Psi(x, x+1) + \mu\Psi(x-1, x), \tag{6}$$

which gives

$$\Psi_{\mathbf{k}}(x_1, x_2) = \mathrm{e}^{\mathrm{i}(k_1 x_1 + k_2 x_2)} + S\mathrm{e}^{\mathrm{i}(k_2 x_1 + k_1 x_2)} \tag{7}$$

with

$$S = -\frac{1 - \lambda \mathrm{e}^{\mathrm{i}k_2} - \mu \mathrm{e}^{-\mathrm{i}k_1}}{1 - \lambda \mathrm{e}^{\mathrm{i}k_1} - \mu \mathrm{e}^{-\mathrm{i}k_2}} \tag{8}$$

and

$$u = (1 - \alpha + \alpha \mathrm{e}^{-\mathrm{i}k_1})(1 - \alpha + \alpha \mathrm{e}^{-\mathrm{i}k_2}). \tag{9}$$

Using these, the conditional probability is obtained as

$$P(\mathbf{x}, t; \mathbf{y}, 0) = \int \frac{\mathrm{d}^2 k}{4\pi^2} \psi_{\mathbf{k}}(\mathbf{x}) \mathrm{e}^{-\mathrm{i}(k_1 y_1 + k_2 y_2)} u^t. \tag{10}$$

One defines the one-particle probabilities as

$$\begin{aligned} P_1(x, t) &:= \sum_{x_2 > x} P(x, x_2, t) \\ P_2(x, t) &:= \sum_{x_1 < x} P(x_1, x, t). \end{aligned} \tag{11}$$

From these,

$$\begin{aligned} P_1(x, t+1) &= (1-\alpha) P_1(x, t) + \alpha P_1(x-1, t) \\ &\quad + \lambda\alpha(1-\alpha)[P(x, x+1, t) - P(x-1, x, t)], \\ P_2(x, t+1) &= (1-\alpha) P_2(x, t) + \alpha P_2(x-1, t) \\ &\quad + \mu\alpha(1-\alpha)[P(x-2, x-1, t) - P(x-1, x, t)]. \end{aligned} \tag{12}$$

Defining

$$\begin{aligned} X_1(t) &:= \sum_x x P_1(x, t), \\ X_2(t) &:= \sum_x x P_2(x, t), \end{aligned} \tag{13}$$

one has

$$\begin{aligned} X_1(t+1) &= X_1(t) + \alpha - \lambda\alpha(1-\alpha) P_r(1, t), \\ X_2(t+1) &= X_2(t) + \alpha + \mu\alpha(1-\alpha) P_r(1, t), \end{aligned} \tag{14}$$

where

$$P_r(x) := \sum_y P(y, y+x). \tag{15}$$

From (10),

$$P_r(x,t) = \int \frac{\mathrm{d}k}{2\pi}[\mathrm{e}^{\mathrm{i}kx} + \mathrm{e}^{-\mathrm{i}k(x-1)}]\mathrm{e}^{-\mathrm{i}k(y_2-y_1)}(1-\alpha+\alpha\mathrm{e}^{-\mathrm{i}k})^t(1-\alpha+\alpha\mathrm{e}^{\mathrm{i}k})^t. \tag{16}$$

A steepest descent calculation shows that for t large and x not large,

$$P_r(x,t) \sim \frac{1}{\sqrt{\pi\alpha(1-\alpha)t}}, \tag{17}$$

from which

$$\begin{aligned} X_1(t) &= X_1(0) + \alpha t - \lambda\left[2\sqrt{\frac{\alpha(1-\alpha)t}{\pi}} + C + \mathrm{o}(1)\right], \\ X_2(t) &= X_2(0) + \alpha t + \mu\left[2\sqrt{\frac{\alpha(1-\alpha)t}{\pi}} + C + \mathrm{o}(1)\right], \end{aligned} \tag{18}$$

or

$$\begin{aligned} X_2(t) - X_1(t) &= X_2(0) - X_1(0) + 2\sqrt{\frac{\alpha(1-\alpha)t}{\pi}} + C + \mathrm{o}(1), \\ \frac{X_1(t)+X_2(t)}{2} &= \frac{X_1(0)+X_2(0)}{2} + \alpha t + (\mu-\lambda)\left[\sqrt{\frac{\alpha(1-\alpha)t}{\pi}} + C + \mathrm{o}(1)\right]. \end{aligned} \tag{19}$$

So the drift rates for large t are

$$\begin{aligned} V_1 &:= \frac{\mathrm{d}X_1}{\mathrm{d}t} = \alpha - \lambda\sqrt{\frac{\alpha(1-\alpha)}{\pi t}}, \\ V_2 &:= \frac{\mathrm{d}X_2}{\mathrm{d}t} = \alpha + \mu\sqrt{\frac{\alpha(1-\alpha)}{\pi t}}, \end{aligned} \tag{20}$$

or

$$\begin{aligned} V_2 - V_1 &= \sqrt{\frac{\alpha(1-\alpha)}{\pi t}}, \\ \frac{V_2+V_1}{2} &= \alpha + \frac{\mu-\lambda}{2}\sqrt{\frac{\alpha(1-\alpha)}{\pi t}}. \end{aligned} \tag{21}$$

X_1 and X_2 are the expectation values of the positions of the first and second particles, respectively, and V_1 and V_2 are their corresponding velocities. As t is discrete, these velocities are defined only when X_is are smooth functions of t, which happens at large times.

The above equations show that the drift velocities of both particles approach α for large times, as at large times the particles are far from each other and effectively do not interact with each other. But the next leading terms in the velocities are negative for the first particle and positive for the second particle, which is expected from the hindering effect of the second particle on the first, and the pushing effect of the first particle on the second. One can see that the results obtained in [14] are recovered, provided that one changes $\alpha(1-\alpha)t$ to t.

4. Concluding remarks

A family of discrete-time asymmetric exclusion processes on a one-dimensional lattice with simultaneous updating was introduced, which is solvable in the sense that the many-particle S-matrix is factorized in terms of two-particle S-matrices. It was shown that this family contains the discrete-time analogues of the so-called simple exclusion process and the drop–push model. The conditional probability and the drift rates of the two-particle sector were obtained, and the results were compared to the known corresponding results for the continuous-time evolution. It was shown that the former coincide with latter at large times, provided that the parameters are suitably redefined.

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